Indian Statistical Institute, Bangalore

B. Math (Hons.) Third Year

Second Semester - Analysis IV

Semestral Exam — Date: May 04, 2018

Maximum marks: 50 — Duration: 3 hours

Section I: Answer any four, each question carries 6 marks

1. If X is a compact metric space and $\mathcal{A}$ is a closed subalgebra of $C_{\mathbb{R}}(X)$ that separates points of X, prove that either $\mathcal{A}$ nowhere vanishes or there is a $x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X) \mid f(x_0) = 0\}$.
2. Let $\mathcal{A}$ be an algebra of real-valued continuous functions on a compact metric space. Let $f \in \mathcal{A}$. Prove that $|f| \in \overline{\mathcal{A}}$. Can $|f| \in \mathcal{A}$? Justify your answer.
3. Discuss Implicit Function Theorem for F at $(2, -1, 2, 1)$ where $F\colon \mathbb{R}^{2+2} \to \mathbb{R}^2$ is given by $F(x, y, u, v) = (x^2 - y^2 - u^3 + v^2 + 4, 2xy + y^2 - 2u^2 + 3v^4 + 8)$.
4. Prove (the three identities in) Parseval's Theorem.
5. Prove that series converges $\sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)x$ to $\frac{1}{2}$ for all $x \in (0, \pi)$.
6. Find total variation of $f(x) = x^3 - 2x^2 + x + 2$ and $g(x) = x^3 - 4x + 5$ on $[0, 1]$.

Section II: Answer any two, each question carries 13 marks

1. (a) Let $E \subset \mathbb{R}^n$ be open and $f\colon E \to \mathbb{R}^n$ be a C^1-map with $f'(x)$ being invertible. Prove that there is neighborhood U of x such that $f(U)$ is open in $\mathbb{R}^n$.

   (b) Let $\mathcal{A}_I$ be set of all polynomials of degree 4 with coefficients from $I \subset \mathbb{R}$. Prove that $\overline{A}_I$ is compact in $C[0, 1]$ if and only if I is bounded (**Marks: 7**).
2. (a) Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $s_n(x)$ be the n-th partial sum of the Fourier series of f at $x \in \mathbb{R}$. If $s(x) = \lim_{t\to 0} \frac{f(x+t)+f(x-t)}{2}$ exists for $x \in [-\pi, \pi]$, prove that $\sigma_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} s_i(x) \to s(x)$.

   (b) Using Fourier series prove that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ (**Marks: 6**).
3. (a) Prove that Fourier series of any 2π-periodic bounded function that is monotonic in $[-\pi, \pi)$ converges (**Marks: 4**).

   (b) Prove that Fourier series of any 2π-periodic function that is monotonic in $(a, b) \subset (-\pi, \pi)$ and Riemann integrable over $[-\pi, \pi]$ converges (**Marks: 3**).

   (c) If f is differentiable such that $f' \in \mathcal{R}[-\pi, \pi]$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(t)|^2 dt \leq 1$. Prove that $|f(x) - s_n(x)| \leq \frac{2}{\sqrt{n}}$ for all $x \in \mathbb{R}$ and $n \geq 1$.